Journal of Engineering Mathematics, Vol. 4, No. 2, April 1970
Wolters-Noordhoff Publishing–Groningen
Printed in the Netherlands

On a Class of Quasi-Conical Supersonic Wings with Curved Subsonic Leading Edges

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(Received September 19, 1969)

SUMMARY

It is shown that for a certain class of wings with subsonic, curved leading edges, in linearized supersonic flow, the perturbation potential can be expanded in terms of functions which are solutions of homogeneous flow problems. If the boundary conditions on the wing are of polynomial form, the homogeneous flow problems are elementary. Some calculations are carried out for flat wings with gothic and ogee planforms under incidence.

1. Introduction

In linearized supersonic flow theory analytic solutions are known for the pressure distribution on wings with straight subsonic leading edges. Solutions of very general and elegant form are given by conical and so-called quasi-conical flow theory. P. Germain [1] has defined homogeneous flows of order n as flows for which the perturbation potential $\varphi(x, y, z)$ is a homogeneous function of order n in the variables x, y and z. Conical flow then, is a flow homogeneous of order one and quasi-conical flow is a flow of higher order of homogeneity or a superposition of such flows for several n. If the leading edges of the wings considered are curved as in the case of gothic and ogee planforms, the flow can in general not be represented as a superposition of homogeneous flows. One possible solution of this problem was given by Adams and Sears [5] in an extension of slender body theory to a not-so-slender body theory by introducing expansions in terms of a slenderness parameter $(\beta S_T)^2$. The first term corresponds with the slender body theory solution. The first and second terms are the not-so-slender body theory solution. The success of the not-so-slender body theory will depend on the value of the slenderness parameter. The calculations carried out by Squire [6] confirm this and indicate further that the range of applicability of the not-so-slender theory depends moreover on the magnitude of the higher streamwise derivatives of the spanwise load, in such a way that this range decreases with increasing magnitude of these derivatives. For moderate values of the slenderness parameter, improvement can be expected by including higher order terms. It is less evident however, that the restrictions with respect to the magnitude of the higher streamwise derivatives should be similarly relieved in the same process. For ogee planforms with a value for $(\beta S_T)^2$ exceeding $(0.3)^2$ and for pitching delta wings with a value of $(\beta S_T)^2$ exceeding, say $(0.4)^2$ the not-so-slender theory is inaccurate. At least one more term is required to obtain better results.

Many planforms which are of interest from a practical point of view differ only slightly from delta planforms for which $(\beta S_T)^2$ will not be small at cruising conditions. It would seem natural therefore to connect the flow around these wings with the flows around delta wings rather than with slender body approximations.

E. Carafoli has given an expression for the disturbance potential in the plane of the wing for a flat wing, with slightly curved leading edges, under incidence and satisfying moreover certain necessary limiting conditions. The parameter occurring in the conical solution is replaced by a function depending only on the streamwise variable x and chosen in such a way that the solution is correct for straight leading edges and also for slender wings with curved leading edges. The functions obtained in this way do not satisfy the differential equation. By doing so one may hope to anticipate certain trends but cannot expect to predict correctly the magnitude of the variations which occur when the wing is not slender.

In the present paper another method for the same problem is suggested. A class of transformations is introduced which transform leading edges of the form $|y| = Ax + Bx^2 + Cx^3 \ldots$ into $|y'| = Ax'$.

Under certain conditions we can satisfy the transformed differential equation and the transformed boundary conditions in terms of homogeneous functions which are solutions of problems in homogeneous flow theory. The order of homogeneity included can be associated with the order of approximation desired. A comparison is made between a first approximation for a flat gothic wing and the corresponding results obtained by Squire [6] and the expression given by Carafoli [7, 8].

2. Formulation of the Problem

In linearized supersonic potential flow the perturbation velocity potential must satisfy

$$\beta^2 \varphi_{xx} - \varphi_{yy} - \varphi_{zz} = 0 \tag{2.1}$$

with $\beta^2 = M^2 - 1$, M being the Mach number of the oncoming flow in the x-direction.

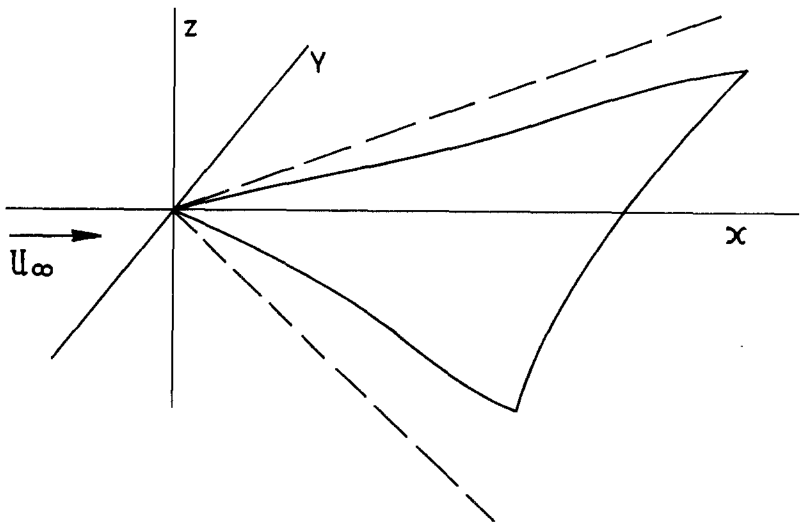

Figure 2.1

Equation (2.1) can be normalized by taking

$$x_1 = \frac{x}{\beta}\,; \quad y_1 = y\,; \quad z_1 = z\,.$$

Dropping the subscripts we have for the perturbation potential

$$\varphi_{xx} - \varphi_{yy} - \varphi_{zz} = 0\,. \tag{2.2}$$

We take $\varphi = 0$ at the Mach cone through the origin. As usual the boundary conditions at the surface of the wing will be applied at the projection of the wing on the plane $z = 0$. The wings considered will be symmetric with respect to the x-axis and the subsonic leading edges will be described by

$$|y| = Ax + Bx^2 + Cx^3 + \ldots\,. \tag{2.3}$$

It can be shown that in linearized airfoil theory the flow around a wing can be decomposed into two parts which can be treated independently. The first can be connected with a source distribution in the plane of the wing and is often referred to as the thickness case; the perturbation potential will be an even function in z. The second part can be connected with a doublet distribution in the plane of the wing and it is often called the lifting case; the perturbation potential is an uneven function in z.

In both cases we can distinguish two types of problems, the direct and inverse problems.

(i) The direct problem consists of finding φ_x at $z=0$, which is proportional to the pressure perturbation at the wing-surface, for given φ_z at $z=0$, the latter being proportional to the local slope of the wing surface in the x-direction.

(ii) The inverse or indirect problem consists of finding φ_z at $z=0$ and thereby a wing-surface that will sustain a prescribed pressure disturbance at $z=0$.

In the direct thickness case and in the indirect lifting case the problem can be reduced to the evaluation of a double integral. The solution of a direct lifting problem and an inverse thickness problem will, in general, require the solution of an integral equation.

3. The Transformations

We introduce the variables x', y' and z' by

$$x' = x + \sum_{p+q+r=2}^{n} a_{pqr} x^p y^q z^r .$$

$$y' = y + \sum_{p+q+r=2}^{n} b_{pqr} x^p y^q z^r .$$

$$z' = z + \sum_{p+q+r=2}^{n} c_{pqr} x^p y^q z^r .$$

$p+q+r=2, 3, \ldots$; p, q and r are non-negative integers.

In order to retain, in some degree, the geometry of the flow-field in the xyz-space, when transforming to the $x'y'z'$-space, we require that x' is invariant when y and z change sign and that y' and z' change sign with y and z respectively without changing their values when z and y, respectively, change sign.

For the transformations this implies:

(i) In x' no odd power of y and z occur.

(ii) In y' only terms with odd powers of y occur, but no odd powers of z.

(iii) In z' only terms with odd powers of z occur, but no odd powers of y.

In the region of interest x and x' will remain positive; in x', y' and z' odd and even powers of x may occur.

Now we can write, with terms up to the third degree included:

$$\begin{aligned} x' &= x + a_{200}x^2 + a_{020}y^2 + a_{002}z^2 + a_{300}x^3 + a_{120}xy^2 + a_{102}xz^2 \ldots \\ y' &= y + b_{110}xy + b_{210}x^2y + b_{030}y^3 + b_{012}yz^2 + \ldots \\ z' &= z + c_{101}xz + c_{201}x^2z + c_{021}y^2z + c_{003}z^3 + \ldots . \end{aligned} \tag{3.1}$$

The Jacobian of this transformation is one at the origin and is positive in a neighbourhood of the origin. In this region we find for the inverse transformation, also up to the third degree terms:

$$\begin{aligned} x &= x' - a_{200}x'^2 - a_{020}y'^2 - a_{002}z'^2 + (2a_{200}^2 - a_{300})x'^3 + \\ &\quad + \{2a_{020}(a_{200} + b_{110}) - a_{120}\}x'y'^2 + \{2a_{002}(a_{200} + c_{101}) - a_{102}\}x'z'^2 \ldots \\ y &= y' - b_{110}x'y' + (b_{110}^2 + a_{200}b_{110} - b_{210})x'^2y' + (a_{020}b_{110} - b_{030})y'^3 + \\ &\quad + (a_{002}b_{110} - b_{012})y'z'^2 + \ldots \\ z &= z' - c_{101}x'z' + (c_{101}^2 + a_{200}c_{101} - c_{201})x'^2z' + (a_{020}c_{101} - c_{021})y'^2z' + \\ &\quad + (a_{002}c_{101} - c_{003})z'^3 + \ldots . \end{aligned} \tag{3.2}$$

An interesting property of these transformations is the possibility to transform leading edges of polynomial form $|y|=Ax+Bx^2+Cx^3+\dots$ into $|y'|=Ax'+B'x'^2+C'x'^3+\dots$ in which $B', C', \dots$ can be made successively equal to zero by making a suitable choice of the parameters in the transformations:

The coefficients of the second degree terms must satisfy

$$B+Ab_{110}=Aa_{200}+A^3a_{020} \tag{3.3}$$

in order to make $B'=0$.

The coefficients of the third degree terms must satisfy

$$C+Bb_{110}+Ab_{210}+A^3b_{030}=2A^2Ba_{020}+Aa_{300}+A^3a_{120} \tag{3.4}$$

in order to make $C'=0$.

We can straighten the leading edge to order p by adapting the coefficients of terms up to degree p in the transformations. In a region not too far from the origin we now write for the leading edges:

$$|y'|=Ax'\,. \tag{3.5}$$

In this paper terms up to the third degree in the transformations will be included. This will make it possible to straighten a leading edge, represented by a third degree polynomial $|y|=Ax+Bx^2+Cx^3$ up to the third order. Thus the complicated wing planform is transformed into a simpler, conical, planform up to the third order. The transformed differential equation for the perturbation potential, however, becomes very complicated:

$$\varphi_{x'x'}g_1+\varphi_{y'y'}g_2+\varphi_{z'z'}g_3+2\varphi_{x'y'}g_4+2\varphi_{x'z'}g_5+2\varphi_{y'z'}g_6+\varphi_{x'}g_7+\varphi_{y'}g_8+\varphi_{z'}g_9=0\,. \tag{3.6}$$

The functions g are given in the appendix.

Equation (3.6) is linear in φ and admits solutions of the form

$$\varphi=(1+\alpha^{(n)}_{100}x'+\alpha^{(n)}_{200}x'^2+\alpha^{(n)}_{020}y'^2+\alpha^{(n)}_{002}z'^2+\dots)\varphi_n \tag{3.7}$$

and hence linear combinations of such solutions for different n, with φ_n denoting a function homogeneous of order n in x', y' and z'.

If we substitute (3.7) in (3.6) and arrange the terms according to increasing orders of homogeneity it is found that the following equations must be satisfied:

for terms homogeneous of order $(n-2)$:

$$\varphi_{nx'x'}-\varphi_{ny'y'}-\varphi_{nz'z'}=0 \tag{3.8}$$

for terms homogeneous of order $(n-1)$:

$$\begin{aligned}&\varphi_{nx'x'}(4a_{200}+\alpha^{(n)}_{100})x'-\varphi_{ny'y'}(2b_{110}+\alpha^{(n)}_{100})x'-\varphi_{nz'z'}(2c_{101}+\alpha^{(n)}_{100})x'+\\&\quad+2\varphi_{nx'y'}(b_{110}-2a_{020})y'+2\varphi_{nx'z'}(c_{101}-2a_{002})z'+\\&\quad+\varphi_{nx'}(2a_{200}-2a_{020}-2a_{002}+2\alpha^{(n)}_{100})=0\end{aligned} \tag{3.9}$$

for terms homogeneous of order (n):

$$\begin{aligned}&\varphi_{nx'x'}h_1+\varphi_{ny'y'}h_2+\varphi_{nz'z'}h_3+2\varphi_{nx'y'}h_4+2\varphi_{nx'z'}h_5+2\varphi_{ny'z'}h_6+\\&\quad+\varphi_{nx'}h_7+\varphi_{ny'}h_8+\varphi_{nz'}h_9+\varphi_nh_{10}=0\,.\end{aligned} \tag{3.10}$$

The functions h are given in the appendix.

If the equation (3.8) for terms of order $(n-2)$ is satisfied, the equation (3.9) for terms of order $(n-1)$ can be satisfied by making a suitable choice of the parameters occurring in the coefficients of the derivatives. φ_n is homogeneous of order n so we have:

$$n\varphi_n=x'\varphi_{nx'}+y'\varphi_{ny'}+z'\varphi_{nz'} \tag{3.11}$$

and by differentiation with respect to x':

$$(n-1)\varphi_{nx'}=x'\varphi_{nx'x'}+y'\varphi_{nx'y'}+z'\varphi_{nx'z'}\,. \tag{3.12}$$

For $n \neq 1$ we can write

$$\varphi_{nx'} = \frac{1}{(n-1)} (x' \varphi_{nx'x'} + y' \varphi_{nx'y'} + z' \varphi_{nx'z'})$$

so that the equation for terms homogeneous of order $(n-1)$ becomes:

$$\begin{aligned} &\varphi_{nx'x'} \left\{ 4a_{200} + \alpha_{100}^{(n)} + \frac{2a_{200} - 2a_{020} - 2a_{002} + 2\alpha_{100}^{(n)}}{(n-1)} \right\} x' + \\ &\quad - \varphi_{ny'y'} (2b_{110} + \alpha_{100}^{(n)}) x' - \varphi_{nz'z'} (2c_{101} + \alpha_{100}^{(n)}) x' + \\ &\quad + 2\varphi_{nx'y'} \left\{ b_{110} - 2a_{020} + \frac{a_{200} - a_{020} - a_{002} + \alpha_{100}^{(n)}}{(n-1)} \right\} y' + \\ &\quad + 2\varphi_{nx'z'} \left\{ c_{101} - 2a_{002} + \frac{a_{200} - a_{020} - a_{002} + \alpha_{100}^{(n)}}{(n-1)} \right\} z' = 0 . \end{aligned}$$

If we now satisfy the equations

$$4a_{200} + \alpha_{100}^{(n)} + 2 \frac{a_{200} - a_{020} - a_{002} + \alpha_{100}^{(n)}}{(n-1)} = 2b_{110} + \alpha_{100}^{(n)} \tag{3.13}$$

$$2b_{110} + \alpha_{100}^{(n)} = 2c_{101} + \alpha_{100}^{(n)} \tag{3.14}$$

$$b_{110} - 2a_{020} + \frac{a_{200} - a_{020} - a_{002} + \alpha_{100}^{(n)}}{(n-1)} = 0 \tag{3.15}$$

$$c_{101} - 2a_{002} + \frac{a_{200} - a_{020} - a_{002} + \alpha_{100}^{(n)}}{(n-1)} = 0 \tag{3.16}$$

it follows that the equation (3.9) for terms homogeneous of order $(n-1)$ is satisfied by φ_n when φ_n is a homogeneous solution of order n to equation (3.8).

The equations (3.13), (3.14), (3.15) and (3.16) are equivalent to

$$a_{020} = a_{002} \tag{3.17}$$

$$a_{200} + a_{020} = b_{110} \tag{3.18}$$

$$b_{110} = c_{101} \tag{3.19}$$

$$\alpha_{100}^{(n)} = (n+1) a_{020} - n a_{200} . \tag{3.20}$$

For $n = 1$ equation (3.12) becomes

$$x' \varphi_{1x'x'} + y' \varphi_{1x'y'} + z' \varphi_{1x'z'} = 0 .$$

It is easily verified that we satisfy equation (3.9) for terms of zero order of homogeneity if we satisfy (3.17), (3.18) and (3.19) while (3.20) becomes

$$\alpha_{100}^{(1)} = 2a_{020} - a_{200}$$

so that equations (3.17), (3.18), (3.19) and (3.20) must be satisfied for all $n = 1, 2, 3, \ldots$. It is important to note that these equations can be satisfied with the parameters of the transformations independents of n. Summarizing it is seen that the six coefficients a_{200}, a_{020}, a_{002}, b_{110}, c_{101} and $\alpha_{100}^{(n)}$ must satisfy five independent equations (3.3), (3.17), (3.18), (3.19) and (3.20).

From (3.3) and (3.18) we obtain

$$a_{020} = \frac{B}{\mathrm{A}^3 - A} \tag{3.21}$$

For terms homogeneous of order n we have:

$$\left.\begin{aligned}(n-1)x'\varphi_{nx'} &= x'^2\varphi_{nx'x'}+x'y'\varphi_{nx'y}+x'z'\varphi_{nx'z'}\\(n-1)y'\varphi_{ny'} &= x'y'\varphi_{nx'y'}+y'^2\varphi_{ny'y'}+y'z'\varphi_{ny'z'}\\(n-1)z'\varphi_{nz'} &= x'z'\varphi_{nx'z'}+y'z'\varphi_{ny'z'}+z'^2\varphi_{nz'z'}\end{aligned}\right\} \tag{3.22}$$

Using relations (3.11) and (3.22) successively the equation (3.10) for terms homogeneous of order (n) is reduced to an equation in which only the six second derivatives of φ_n occur for $n\neq 1$.

For $n=1$, φ, can be eliminated by (3.11) for $n=1$ and the cross-derivatives by:

$$\left.\begin{aligned}2x'y'\varphi_{1x'y'} &= z'^2\varphi_{1z'z'}-x'^2\varphi_{1x'x'}-y'^2\varphi_{1y'y'}\\2x'z'\varphi_{1x'z'} &= y'^2\varphi_{1y'y'}-z'^2\varphi_{1z'z'}-x'^2\varphi_{1x'x'}\\2y'z'\varphi_{1y'z'} &= x'^2\varphi_{1x'x'}-y'^2\varphi_{1y'y'}-z'^2\varphi_{1z'z'}\,.\end{aligned}\right\} \tag{3.23}$$

If φ_n is a solution of equation (3.8) the equation (3.10) for terms homogeneous of order n is also satisfied by φ_n provided six equations independent of n are satisfied, for the nine coefficients of the third degree terms in the transformations:

$$\left.\begin{aligned}a_{102}-b_{012} &= a_{020}^2+a_{200}a_{020}\\a_{120}-b_{030} &= a_{020}^2+a_{200}a_{020}\\b_{210}-a_{300} &= a_{020}^2+a_{200}a_{020}\\b_{210} &= c_{201}\\b_{030} &= c_{021}\\b_{012} &= c_{003}\end{aligned}\right\} \tag{3.24}$$

and three equations for the three coefficients $\alpha_{200}^{(n)}$, $\alpha_{020}^{(n)}$ and $\alpha_{002}^{(n)}$, which for $n=1$ simplify to:

$$\left.\begin{aligned}\alpha_{200}^{(1)} &= 3a_{020}^2-4a_{200}a_{020}+2a_{200}^2-a_{300}\\2\alpha_{020}^{(1)} &= 2a_{200}a_{020}-a_{020}^2-2b_{030}\\2\alpha_{002}^{(1)} &= 2a_{200}a_{020}-a_{020}^2-2c_{003}\end{aligned}\right\} \tag{3.25}$$

From equation (3.4) and the second, third and fifth equations of (3.24) it follows that A, B and C must satisfy

$$C=\frac{2AB^2}{A^2-1}\,. \tag{3.26}$$

By restricting ourselves to solutions of the form (3.7) we restrict ourselves to leading edges of polynomial form for which the first three coefficients satisfy (3.26). For subsonic leading edges we have $0<A<1$ so that C will not be positive while B can be positive, negative and zero. $B=0$ gives $C=0$. Negative B gives gothic planforms if we include second degree terms in the leading edge equation and in the transformations and also if we include third degree terms. For positive B we find concave planforms if we include second degree terms and concave and ogee planforms if we include third degree terms. It must be assumed that we are not too far from the origin so that higher degree terms can indeed be neglected with respect to those retained.

The degree of the terms included can also be connected with a desired order of approximation. If we take B small of order ε, the coefficients of second degree terms in the transformations are small of order ε. C and the coefficients of third degree terms in the transformations will be small of order $(\varepsilon)^2$. From the transformed differential equation (3.6) it is noticed that in the coefficients of the second derivatives terms of degree one are of order ε and terms of degree two are of order $(\varepsilon)^2$. In the coefficients of first derivatives, terms of degree zero are of order ε and terms of degree one are of order $(\varepsilon)^2$.

Arranging the terms in the transformed equation according to orders of ε leads to equations for the coefficients in the transformations which are the same as those obtained when we substitute solutions of the form (3.7) and arrange according to orders of homogeneity.

4. The Boundary Conditions

It is easily verified that equations (3.18), (3.19) and (3.24) imply that the characteristic cone $x^2=y^2+z^2$ transforms into $x'^2=y'^2+z'^2$ up to the order considered so that $\varphi=0$ at $x^2=y^2+z^2$ becomes $\varphi=0$ at $x'^2=y'^2+z'^2$.

Outside the projection of the wing at $z=0$ we have $\varphi_x=0$ in the lifting case because φ_x is continuous and uneven in z; in the thickness-case we have $\varphi_z=0$ at $z=0$ outside the projection of the wing.

In a direct problem φ_z is given at the projection of the wing on $z=0$.

$$\varphi_{z'} = \varphi_x \frac{\partial x}{\partial z'} + \varphi_y \frac{\partial y}{\partial z'} + \varphi_z \frac{\partial z}{\partial z'} \tag{4.1}$$

For $z=0$ we have $z'=0$, $\partial x/\partial z'=0$ and $\partial y/\partial z'=0$ so:

$$\varphi_{z'(z'=0)} = \varphi_{z(z=0)}\{1-c_{101}x'+(c_{101}^2+a_{200}c_{101}-c_{201})x'^2+(a_{020}c_{101}-c_{021})y'^2+\dots\} \tag{4.2}$$

In an inverse problem we know φ_x at $z=0$

$$\varphi_{x'} = \varphi_x \frac{\partial x}{\partial x'} + \varphi_y \frac{\partial y}{\partial x'} + \varphi_z \frac{\partial z}{\partial x'}\,. \tag{4.3}$$

At $z=0$ we have $z'=0$ and $\partial z/\partial x'=0$ so we can write:

$$\begin{aligned}\varphi_{x'(z'=0)} = {}& \varphi_{x(z=0)}\{1-2a_{200}x'+3(2a_{200}^2-a_{300})x'^2+[2a_{020}(a_{200}+b_{110})-a_{120}]y'^2+\dots\}\\ &+\varphi_{y(z=0)}\{-b_{110}y'+2(b_{110}^2+a_{200}b_{110}-b_{210}]x'y'\dots\}\,.\end{aligned} \tag{4.4}$$

In a lifting case φ_x is zero at $z=0$ outside the projection of the wing so we can write:

$$\varphi(z=0) = \int_{\text{leading edge}}^{x} \varphi_x dx \tag{4.5}$$

and we can calculate

$$\varphi_{y(z=0)} = \frac{\partial}{\partial y}\int_{\text{leading edge}}^{x} \varphi_x dx\,. \tag{4.6}$$

In the inverse thickness case φ and φ_y do not follow directly from $\varphi_{x(z=0)}$. From (4.2) one sees that if $\varphi_{z(z=0)}$ can be decomposed into terms with different orders of homogeneity this will also be the case for $\varphi_{z'(z'=0)}$. A term homogeneous of order p in $\varphi_{z(z=0)}$ in the variables x and y will give terms homogeneous of orders p, $p+1$, $p+2$, ... for $\varphi_{z'(z'=0)}$ in the variables x' and y'.

From (4.4) one sees that if $\varphi_{x(z=0)}$ can be decomposed into terms with different orders of homogeneity this will also be the case for $\varphi_{x'(z'=0)}$ provided $\varphi_{y(z=0)}$ can also be decomposed in this way (or if we take $b_{110}=0$ and $b_{210}=0$).

5. Applications

We shall now carry out some calculations, first retaining only the first and second degree terms in the transformations, for a flat wing under incidence with leading edges $|y|=Ax+Bx^2$. The boundary conditions on the wing become:

$$\varphi_{z'(z'=0)} = \varphi_{z(z=0)}(1-c_{101}x')\,.$$

We write $\varphi_{z(z=0)} = -U_\infty\alpha = -w_o$ so that

$$\varphi_{z'(z'=0)} = -w_o + w_o c_{101} x' . \tag{5.1}$$

The first part is homogeneous of order zero, the second part is homogeneous of order one.
Up to the order considered we can write for the solution:

$$\varphi = \varphi_1 (1 + \alpha_{100}^{(1)} x') + \varphi_2 . \tag{5.2}$$

In this expression φ_1 and φ_2 are homogeneous of degree one and two respectively and both are solutions of equation (3.8). From equation (3.20) we have for $n=1$:

$$\alpha_{100}^{(1)} = 2a_{020} - a_{200} . \tag{5.3}$$

From equation (5.2) follows

$$\varphi_{z'} = \varphi_{1z'} (1 + \alpha_{100}^{(1)} x') + \varphi_{2z'} . \tag{5.4}$$

From (3.18), (3.19), (5.1), (5.3) and (5.4) we then find:

$$\varphi_{1z'(z'=0)} = -w_o \tag{5.6}$$

and

$$\varphi_{2z'(z'=0)} = 3a_{020} w_o x' . \tag{5.7}$$

The problem for φ_1 is equivalent to the problem of the flow around a flat delta wing under incidence. Its solution is well known:

$$\varphi_{1(z'=+0)} = \frac{w_o}{E'} \sqrt{A^2 x'^2 - y'^2} \tag{5.8}$$

with E' representing the complete elliptic integral of the second kind with modulus $\sqrt{1-A^2}$.
The problem for φ_2 is equivalent to the problem of the flow around a delta wing due to a pitching motion or the problem of a flow around a wing with a parabolic warp in the x' direction. φ_2 can be calculated by the method indicated by Fenain [3].
The solution is:

$$\varphi_{2(z'=+0)} = \frac{-3\, w_o (1-A^2) a_{020}}{E'(1-2A^2) + A^2 K'} x' \sqrt{A^2 x'^2 - y'^2} \tag{5.9}$$

in which K' is the complete elliptic integral of the first kind with modulus $\sqrt{1-A^2}$.
For the solution on the wing we can now write

$$\varphi(z'=+0) = \left[\frac{w_o}{E'} \{1 + (2a_{020} - a_{200}) x'\} - \frac{3(1-A^2)\, w_o a_{020}}{E'(1-2A^2) + A^2 K'} x' \right] \sqrt{A^2 x'^2 - y'^2} . \tag{5.10}$$

For convenience the parameter a_{200}, which is still free, is now so chosen that $(A^2 x'^2 - y'^2)$ transforms into $(Ax + Bx^2)^2 - y^2$ up to the order considered. By equating terms up to the third degree we find

$$a_{200} = \frac{B}{A} . \tag{5.11}$$

The perturbation potential on the wing now becomes:

$$\varphi(z=+0) = w_o \sqrt{(Ax+Bx^2)^2 - y^2} \left[\frac{1}{E'} + Bx \frac{2E'A^3 - 4E'A + (3-A^2)AK'}{E'(A^2-1)\{E'(1-2A^2) + A^2 K'\}} \right] . \tag{5.12}$$

We have $\lim_{A\to 0} E' = 1$ and $\lim_{A\to 0} AK' = 0$ so that for small A and not too large Bx we can write, up to the order considered:

$$\varphi = w_o \sqrt{(Ax+Bx^2)^2 - y^2} . \tag{5.13}$$

This is the well known slender body solution for a flat wing with leading edges $|y| = Ax + Bx^2$. This solution is accurate for small A and small Bx. In solution (5.12) the restriction with respect to A is removed. Carafoli's corresponding solution for the case in which no restriction is imposed on A except $0 < A < 1$ is

$$\varphi(z = +0) = \frac{W_o}{E'} \sqrt{(Ax+Bx^2)^2 - y^2} . \tag{5.14}$$

$$\text{with } E' = \int_0^{\pi/2} \sqrt{1 - \{1-(A+Bx)^2\} \sin^2 \varphi} \, d\varphi .$$

In the elliptic integral occurring in the conical solution the constant A has been, rather arbitrarily, replaced by the variable $(A + Bx)$. The expression $\sqrt{(Ax+Bx^2)^2 - y^2}$ is such that this part is the correct solution in the slender body case and in the conical case, i.e. $B = 0$. The elliptic integral is the correct coefficient for a straight leading edge and near the origin for a curved leading edge, when $Bx \ll A$.

For small Bx we can write

$$\frac{1}{E'(\sqrt{1-(A+Bx)^2})} = \frac{1}{E'(\sqrt{1-A^2})} + Bx \frac{A}{A^2-1} \frac{K'(\sqrt{1-A^2}) - E'(\sqrt{1-A^2})}{\{E'(\sqrt{1-A^2})\}^2} =$$

$$= \frac{1}{E'(\sqrt{1-A^2})} + Bxp .$$

The factor p has been plotted against A and is compared to the corresponding factor as calculated from (5.12). The factors are the same for $A = 0$ and $A = 1$ but for $A = 0.1$ they differ by a factor 2.5.

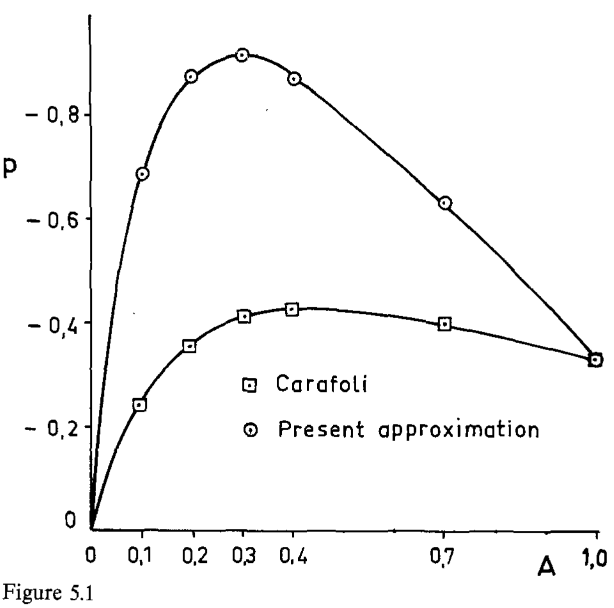

Figure 5.1

The solution (5.12) is easily generalized to other Mach numbers. With $\beta = \sqrt{M^2 - 1}$ we find:

$$\varphi(z = +0) = W_o \sqrt{(Ax+Bx^2)^2 - y^2} \left[\frac{1}{E'} + B\beta x \frac{2E'\beta^3 A^3 - 4E'A\beta + (3 - A^2\beta^2) A\beta K'}{(A^2\beta^2 - 1) E' \{E'(1 - 2A^2\beta^2) + A^2\beta^2 K'\}} \right] \tag{5.15}$$

in which E' and K' now have modulus $\sqrt{1 - A^2\beta^2}$.

The coefficient

$$\frac{2E'\beta^3A^3-4E'A\beta+(3-A^2\beta^2)A\beta K'}{(A^2\beta^2-1)E'\{E'(1-2A^2\beta^2)+A^2\beta^2K'\}}$$

is found from fig. 5.1 by taking $A\beta$ for A. The pressure perturbation follows from $p'=-\rho U\varphi_x$.

Squire [6] has plotted the chordwise variation of the spanwise load on several wings for different Mach-numbers as calculated by the not-so-slender body theory.

To compare the results of the not-so-slender body theory, of Carafoli's approximation and of the present first approximation it seems useful to plot the chordwise variation of load for a gothic wing with leading edges $y=S_Tg(x)$. For $\beta=1$ we have $S_T=0.4$, $g(x)=x(2-x)$ (fig. 5 of reference [6]). In our notation we have

$$|y|=0.8x-0.4x^2\,.$$

For $\beta\to 0$ the three solutions tend to the slender body approximation. We have plotted

$$\frac{L(x)}{2\pi\alpha}$$

with $L(x)=\int_{-(Ax+Bx^2)}^{Ax+Bx^2}\frac{4}{U_\infty}\varphi_x(x,y+0)dy\,.$

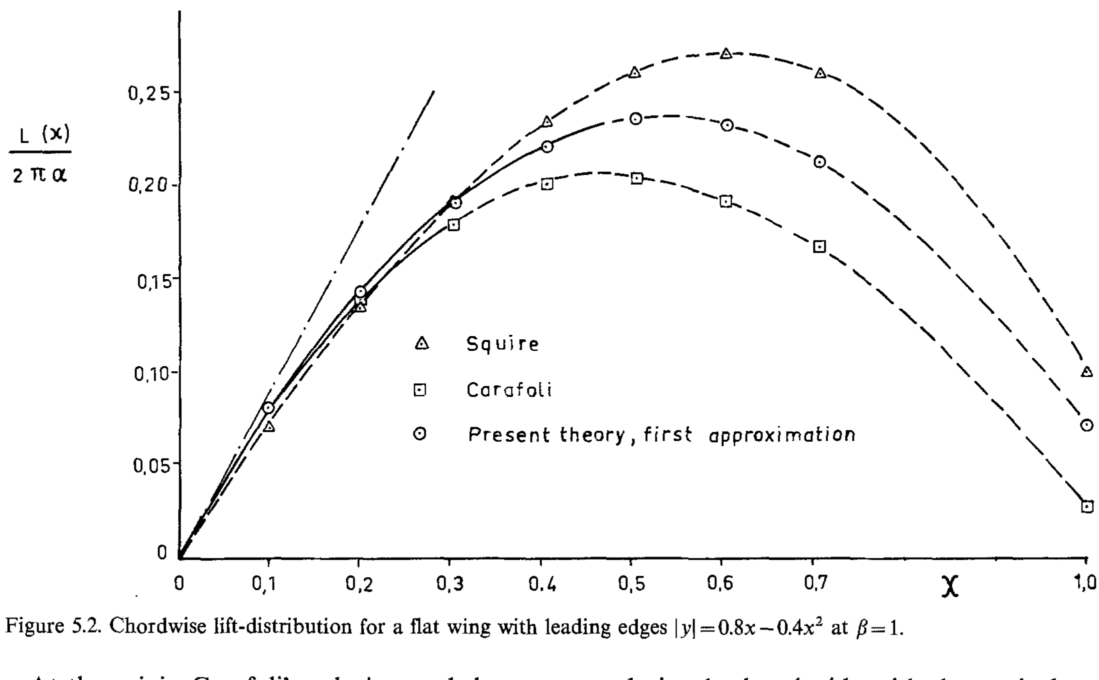

Figure 5.2. Chordwise lift-distribution for a flat wing with leading edges $|y|=0.8x-0.4x^2$ at $\beta=1$.

At the origin Carafoli's solution and the present solution both coincide with the conical solution for a wing with leading edges $|y|=0.8x$. The not-so-slender theory underestimates the lift near the origin with 14%, which explains the difference in slope of the lift distribution curves near the origin.

Comparison with fig. 12 of reference [6] indicates that the present approximation gives an accurate prediction of the experimental total lift. Carafoli's solution underestimates the lift, as compared to the experimental value, by some 15%. The not-so-slender theory overestimates the lift by 8%.

It should be remarked that Carafoli's solution and the present approximation cannot be expected to be very accurate beyond say $x=0.3$ because the leading edge will differ too much from the tangent through the origin. The general trends of the three solutions, however, are very similar over the whole wing.

One can obtain a second order solution by including third degree terms in the transformations and in the equation of the leading edges. We will illustrate this in the following calculation.

Let us consider a flat wing under incidence with leading edges

$$|y| = Ax+Bx^2+Cx^3 .$$

Straightening the leading edges up to the third order gives equations (3.3) and (3.4).

The solution is written (up to the order considered):

$$\varphi = \varphi_1(1+\alpha^{(1)}_{100}x'+\alpha^{(1)}_{200}x'^2+\alpha^{(1)}_{020}y'^2+\alpha^{(1)}_{002}z'^2)+\varphi_2(1+\alpha^{(2)}_{100}x')+\varphi_3 \tag{5.16}$$

in which φ_1, φ_2 and φ_3 are solutions to equation (3.8) and are homogeneous functions of orders one, two and three respectively, in the variables x', y' and z'. The coefficients in the transformations must satisfy equations (3.17), (3.18), (3.19) and (3.24). The coefficients $\alpha^{(1)}_{100}$ and $\alpha^{(2)}_{100}$ follow from (3.20) for $n=1$ and $n=2$ respectively. $\alpha^{(1)}_{200}$, $\alpha^{(1)}_{020}$ and $\alpha^{(1)}_{002}$ follow from (3.25). A, B and C must satisfy equation (3.26).

The boundary conditions at the projection of the wing ($z'=0$) for φ_1, φ_2 and φ_3 become:

$$\varphi_{1z'} = -w_o$$

$$\varphi_{2z'} = 3a_{020}\,w_o x'$$

$$\varphi_{3z'} = -w_o(6a^2_{020}x'^2+\tfrac{3}{2}a^2_{020}y'^2)\,.$$

We require that $A^2x'^2-y'^2$ transforms into $(Ax+Bx^2+Cx^3)^2-y^2$ up to the order considered; this gives:

$$a_{020} = \frac{B}{A^3-A}\,, \quad a_{200} = \frac{B}{A}\,.$$

$$a_{300} = \frac{2B^2}{A^2-1}\,, \quad a_{120} = \frac{3B^2}{2(A^2-1)^2}\,.$$

The solutions φ_1 and φ_2 are the same as in the previous example. The solution for φ_3 at $z'=0$ can be calculated by the method indicated by Fenain [3].

The solution can be written:

$$\varphi_{3(z'=0)} = w_o(C_{20}x'^2+C_{02}y'^2)\sqrt{A^2x'^2-y'^2} \tag{5.17}$$

with

$$\left.\begin{aligned} C_{20} &= \frac{-B^2}{2A^2(A^2-1)^2}\,\frac{(-24+38A^2-10A^4)E'+(24-36A^2)A^2K'}{(4-19A^2+4A^4)E'^2+8A^2(1+A^2)E'K'-5A^4K'^2} \\ C_{02} &= \frac{-B^2}{2A^2(A^2-1)^2}\,\frac{(-96+244A^2-20A^4-12A^6)E'+(24-18A^2+6A^4)A^4K'}{A^2\{(4-19A^2+4A^4)E'^2+8A^2(1+A^2)E'K'-5A^4K'^2\}} \end{aligned}\right\} \tag{5.18}$$

In the original coordinate system, on the upperside of the wing, we find for the disturbance potential:

$$\varphi = w_o\sqrt{(Ax+Bx^2+Cx^3)^2-y^2}\left\{C_{00}+C_{10}x+\left(C_{20}+\frac{B}{A}C_{10}\right)x^2+ \right.$$

$$\left. +\left(C_{02}+\frac{B}{A^3-A}C_{10}\right)y^2\right\} \tag{5.19}$$

with $C_{00} = \dfrac{1}{E'}$, $C_{10} = B\dfrac{2E'A^3-4E'A+(3-A^2)AK'}{(A^2-1)E'\{E'(1-2A^2)+A^2K'\}}$

and C_{20} and C_{02} from (5.18).

By taking the derivative of φ from (5.19) with respect to x we find the pressure distribution on the wing ($p'=-\rho U\varphi_x$).

6. Concluding Remarks

We have carried out some calculations for flat wings. In the case of warped wings, the flat plate solutions occur as the incidence-dependent part of the solution. From equation (4.2) it is seen that if $\varphi_z(z=0)$ is of polynomial form in x and y, $\varphi_{z'}(z'=0)$ will be of polynomial form in x' and y'. In this case the problem is also reduced to the solution of elementary problems in homogeneous flow theory, which are formally equivalent to the problems we have to solve for a flat wing. A term of degree $(n-1)$ will lead to elementary problems of orders $n, n+1, n+2, \ldots$. If $\varphi_z(z=0)$ includes terms of the form $x^{(n-1)}f(y/x)$, $\varphi_{z'}(z'=0)$ will include terms of the form $x'^{(n-1+p)}g_p(y'/x')$, with $p=0, 1, 2, \ldots$ and such boundary conditions lead to non-elementary homogeneous flow problems of order $(n+p)$. From the expressions for the disturbance potential it is easy to calculate the pressure distribution on a class of flat wings. From the strength of the singularity at the leading edge we find at once the variation of the suction force along the leading edge.

Some further calculations seem to indicate that the present results remain probably accurate at a larger distance from the origin in case B is negative (gothic planforms) than in case B is positive. We intend to come back to this point in a future communication.

Acknowledgement

The author expresses his gratitude to prof. dr. ir. J. A. Steketee, under whose supervision this work has been done, for the many helpful discussions during the preparation of the manuscript.

Appendix

The Functions g and h

The functions g in formula (3.6) are, up to the order considered:

$$g_1 = \left(\frac{\partial x'}{\partial x}\right)^2 - \left(\frac{\partial x'}{\partial y}\right)^2 - \left(\frac{\partial x'}{\partial z}\right)^2 = 1 + 4a_{200}x' + 6a_{300}x'^2 + (2a_{120} - 4a_{020}^2 - 4a_{200}a_{020})y'^2 + (2a_{102} - 4a_{002}^2 - 4a_{200}a_{002})z'^2 \ldots$$

$$g_2 = \left(\frac{\partial y'}{\partial x}\right)^2 - \left(\frac{\partial y'}{\partial y}\right)^2 - \left(\frac{\partial y'}{\partial z}\right)^2 = -1 - 2b_{110}x' + (2a_{200}b_{110} - b_{110}^2 - 2b_{210})x'^2 + (2a_{020}b_{110} + b_{110}^2 - 6b_{030})y'^2 + (2a_{002}b_{110} - 2b_{012})z'^2 \ldots$$

$$g_3 = \left(\frac{\partial z'}{\partial x}\right)^2 - \left(\frac{\partial z'}{\partial y}\right)^2 - \left(\frac{\partial z'}{\partial z}\right)^2 = -1 - 2c_{101}x' + (2a_{200}c_{101} - 2c_{201} - c_{101}^2)x'^2 + (2a_{020}c_{101} - 2c_{021})y'^2 + (2a_{002}c_{101} + c_{101}^2 - 6c_{003})z'^2 \ldots$$

$$g_4 = \frac{\partial x'}{\partial x}\cdot\frac{\partial y'}{\partial x} - \frac{\partial x'}{\partial y}\cdot\frac{\partial y'}{\partial y} - \frac{\partial x'}{\partial z}\cdot\frac{\partial y'}{\partial z} = (b_{110} - 2a_{020})y' + (2a_{200}b_{110} + 2b_{210} - 2a_{120} - b_{110}^2)x'y' \ldots$$

$$g_5 = \frac{\partial x'}{\partial x}\cdot\frac{\partial z'}{\partial x} - \frac{\partial x'}{\partial y}\cdot\frac{\partial z'}{\partial y} - \frac{\partial x'}{\partial z}\cdot\frac{\partial z'}{\partial z} = (c_{101} - 2a_{002})z' + (2a_{200}c_{101} + 2c_{201} - 2a_{102} - c_{101}^2)x'z' \ldots$$

$$g_6 = \frac{\partial y'}{\partial x}\cdot\frac{\partial z'}{\partial x} - \frac{\partial y'}{\partial y}\cdot\frac{\partial z'}{\partial y} - \frac{\partial y'}{\partial z}\cdot\frac{\partial z'}{\partial z} = (b_{110}c_{101} - 2c_{021} - 2b_{012})y'z' \ldots$$

$$g_7 = \frac{\partial^2 x'}{\partial x^2} - \frac{\partial^2 x'}{\partial y^2} - \frac{\partial^2 x'}{\partial z^2} = 2(a_{200} - a_{020} - a_{002}) + (6a_{300} - 2a_{120} - 2a_{102})x' \ldots$$

$$g_8 = \frac{\partial^2 y'}{\partial x^2} - \frac{\partial^2 y'}{\partial y^2} - \frac{\partial^2 y'}{\partial z^2} = (2b_{210} - 6b_{030} - 2b_{012})y' \ldots$$

$$g_9 = \frac{\partial^2 z'}{\partial x^2} - \frac{\partial^2 z'}{\partial y^2} - \frac{\partial^2 z'}{\partial z^2} = (2c_{201} - 2c_{021} - 6c_{003})z' \ldots$$

The functions h in formula (3.10) are:

$$h_1 = (6a_{300} + \alpha_{200}^{(n)} + 4a_{200}\alpha_{100}^{(n)})x'^2 + (2a_{120} - 4a_{020}^2 - 4a_{200}a_{020} + \alpha_{020}^{(n)})y'^2 + \\ + (2a_{102} - 4a_{002}^2 - 4a_{200}a_{002} + \alpha_{002}^{(n)})z'^2 .$$

$$h_2 = (2a_{200}b_{110} - b_{110}^2 - 2b_{210} - 2b_{110}\alpha_{100}^{(n)} - \alpha_{200}^{(n)})x'^2 + \\ + (2a_{020}b_{110} + b_{110}^2 - 6b_{030} - \alpha_{020}^{(n)})y'^2 + (2a_{002}b_{110} - 2b_{012} - \alpha_{002}^{(n)})z'^2 .$$

$$h_3 = (2a_{200}c_{101} - c_{101}^2 - 2c_{201} - 2c_{101}\alpha_{100}^{(n)} - \alpha_{200}^{(n)})x'^2 + \\ + (2a_{020}c_{101} - 2c_{021} - \alpha_{020}^{(n)})y'^2 + (2a_{002}c_{101} + c_{101}^2 - \alpha_{002}^{(n)} - 6c_{003})z'^2 .$$

$$h_4 = (2a_{200}b_{110} + 2b_{210} - 2a_{120} - b_{110}^2 + b_{110}\alpha_{100}^{(n)} - 2a_{020}\alpha_{100}^{(n)})x'y' .$$

$$h_5 = (2a_{200}c_{101} + 2c_{201} - 2a_{102} - c_{101}^2 + c_{101}\alpha_{100}^{(n)} - 2a_{002}\alpha_{100}^{(n)})x'z' .$$

$$h_6 = (b_{110}c_{101} - 2c_{021} - 2b_{012})y'z' .$$

$$h_7 = (6a_{300} - 2a_{120} - 2a_{102} + 10a_{200}\alpha_{100}^{(n)} + 4\alpha_{200}^{(n)} - 2a_{020}\alpha_{100}^{(n)} - 2a_{002}\alpha_{100}^{(n)})x' .$$

$$h_8 = (2b_{210} - 6b_{030} - 2b_{012} - 4\alpha_{020}^{(n)} + 2b_{110}\alpha_{100}^{(n)} - 4a_{020}\alpha_{100}^{(n)})y' .$$

$$h_9 = (2c_{201} - 2c_{021} - 6c_{003} - 4\alpha_{002}^{(n)} + 2c_{101}\alpha_{100}^{(n)} - 4a_{002}\alpha_{100}^{(n)})z' .$$

$$h_{10} = (2\alpha_{200}^{(n)} - 2\alpha_{020}^{(n)} - 2\alpha_{002}^{(n)} + 2a_{200}\alpha_{100}^{(n)} - 2a_{020}\alpha_{100}^{(n)} - 2a_{002}\alpha_{100}^{(n)}) .$$

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